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1995 J. Phys. A: Math. Gen. 28 2737

(http://iopscience.iop.org/0305-4470/28/10/007)

J. Phys. A: Math. Gen. 28 (1995) 2737–2743. Printed in the UK

# The spin-glass model on hierarchical lattices*

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Received 10 February 1995

**Abstract.** We study a frustrated spin-glass model on recursive diamond-shaped lattices. We prove the existence of a mean-field-type phase transition and analyse the high- and low-temperature regions.

## 1. Introduction

In this paper, we analyse the critical behaviour of a frustrated spin glass defined on a hierarchical diamond lattice. Hierarchical-type lattices have been introduced in several contexts (for instance, in the study of percolation, the random-field Ising model, random surfaces, random resistor networks, etc) because the renormalization transformation is exact. The definition of spin-glass models in such lattices provides a non-trivial framework between the infinite-range version of the mean-field theory and the heavily debated short-range spin glass in finite dimensions. The problem was investigated several years ago in [2, 3, 11] and rigorously in [6] by means of renormalization-group techniques. In [2, 3], chaotic renormalization-group trajectories were obtained for the first time and have been related to an intermediate-range chaotic spin-glass order. In [11, 6], the main object of study was the evolution of the probability distribution of the random couplings under a change of scale. In particular, the existence of a non-trivial fixed point corresponding to the spin-glass transition has been proved. In terms of spin observables, this transition is expressed by a non-vanishing Edwards–Anderson parameter at low temperature. This analysis was extended in [17] and the existence of another fixed point corresponding to a mixed ferromagnetic (antiferromagnetic) spin-glass phase was proved.

In the following, we investigate this model using a different approach which allows a complete characterization of the phase transition (for some other applications see [7, 13–16]). Instead of addressing the question of the evolution of the effective couplings as a function of the size of the lattice, we formulate the model in terms of martingales depending on the temperature and look at their behaviour at the thermodynamic limit. It turns out that in many interesting cases these martingales are not regular at the critical temperature. We can thus relate the phase transition to the singular behaviour of the corresponding martingale. This approach was developed using the thermodynamic formalism [18] in several situations [7, 16, 13, 14]. One of the main advantages is that we can treat long- and short-range interactions. Let us also mention that a mean-field-type transition was studied in the case of

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a short-range spin glass on the Bethe lattice [5]. Here also, the definition on the model on a tree allows the study of a recursion relation. While we do not expect the short-range spin glass in finite dimensions to have a mean-field behaviour (a first rigorous insight is given in [14]), the rigorous study of mean-field models should provide some qualitative aspects of the phase transition.

In section 2 we reformulate the model. In section 3 we calculate the critical temperature and analyse the high-temperature region. Indeed, we show the existence and the strong self-averaging property of the free energy and the uniqueness of the Gibbs state. An explicit formula for the free energy at low temperatures is given in section 4. As we shall see, the spin glass studied in this paper undergoes a mean-field-type phase transition (in fact, very similar to that of the random energy model [9]).

## 2. The model

In this section, we express the model in terms of a random process. Let us first recall the definition of a spin glass on the hierarchical diamond lattice.

Let $p$ be an integer $\geqslant 2$. We can define, at the first step, the lattice $\Lambda_0$ by two sites and a bond $b^0$. Having defined the lattice $\Lambda_{N-1}$ at step $N$, we construct the lattice $\Lambda_N$ by replacing each bond $b_i^{(N-1)}$, $i = 1, \ldots, 2p$ by $p$ sites and $2p$ bonds $b_i^{(N)}$ relating each added site to the endpoints of the bond $b_i^{(N-1)}$. Thus, the lattice $\Lambda_N$ has $(2p)^N$ bonds $b_i^{(N)}$.

To each site we associate an Ising spin space $\{-1, 1\}$. The set of all possible configurations is then given by the product space $X_N = \{-1, 1\}^{\Lambda_N}$ equipped with the counting measure. For notational convenience, let us note by $s$ and $s'$ the spins associated with the two sites of the lattice $\Lambda_0$ and by $s_i^{(j)}$, $i = 1, \ldots, 2p$, $j = 1, \ldots, N$, the spins associated with the sites of $\Lambda_N$. We shall also use the symbol $s_{b_i^{(j)}}$ to denote the product of the values of the spin variables at the endpoints of the bond $b_i^{(j)}$. In figure 1 we present the first and the second step of the build-up of the diamond lattice.

Let $J_0$ be a Gaussian variable defined on the probability space $(\Omega, \mathcal{F}, P)$ of mean zero and finite variance $EJ_0^2$ (without loss of generality we shall assume in the following that $EJ_0^2 = 1$). To each bond $b_i^{(j)}$, $i = 1, \ldots, 2p$, $j = 1, \ldots, N$, we now associate the random couplings $J_{b_i^{(j)}}$ having common distribution with $J_0$.

The Hamiltonian of the model in $\Lambda_N$ is given formally by

$$H_{\Lambda_N}(s_b) = \sum_{b^{(N)}} J_{b^{(N)}} s_{b^{(N)}} .$$

We shall also use the abbreviation

$$h_{\Lambda_N}(s_b) = \mathrm{e}^{-\beta H_{\Lambda_N}(s)}$$

for the Boltzmann factor.

We see that the model is parametrized by $p$ and the law of $J_0$. Since $J_0$ is Gaussian with zero mean, the only relevant parameter of its law is its variance. Notice, however,

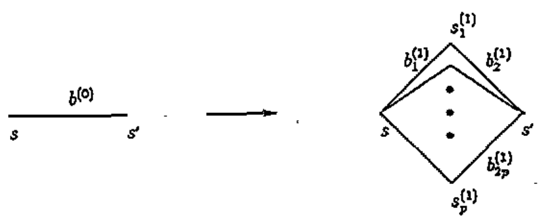


**Figure 1.** Constructing the lattice $\Lambda_1$.

that only the combination $\beta J_0$ enters in the definition of the model; therefore, the model is actually characterized by $p$ and $\beta^2$, where $\beta > 0$ is the inverse temperature.

We also recall the definitions of the thermodynamic quantities to be studied in the following sections. The specific free energy is given by

$$-\beta f_{\Lambda_N}(\beta) = |\Lambda_N|^{-1} \log Z_{\Lambda_N}(\beta)$$

where $Z_{\Lambda_N}(\beta) = \sum_{s_b} h_{\Lambda_N}(s_b)$ is the partition function. Finally, the Gibbs distribution $\nu$ has a density with respect to the product measure $\mathrm{d}s$

$$\nu_{\Lambda_N,\beta}(\mathrm{d}s) = Z_{\Lambda_N}^{-1}(\beta) h_{\Lambda_N}(s_b)\, \mathrm{d}s\,.$$

It is worth noticing that the definition of the previous quantities in terms of the bond configurations $\sigma_b$ is equivalent to the conventional definition in terms of the spin configurations $\sigma$.

Let $G'_{\Lambda_N}(\beta)$ be the sequence of the random variables

$$G'_{\Lambda_N}(\beta) = \frac{\exp(\beta \sum_{b^{(N)}} J_{b^{(N)}} s_{b^{(N)}})}{\exp(\frac{\beta^2}{2}(2p)^N)}\,.$$

If $\mathcal{F}_l$ is the $\sigma$-field generated by the variables $J_{b_i^{(l)}}$, we have the following:

*Proposition. 2.1.* For every $\beta > 0$, the sequence $G'_{\Lambda_N}(\beta, s_b)$ is a positive integrable martingale with $G'_{\Lambda_0} = 1$. Moreover, $G'_{\Lambda_N}(\beta, s_b)$ converges a.s. for $N \to \infty$ to the integrable random variable $G'_\infty(\beta, s_b)$ with $EG'_\infty(\beta, s_b) \leqslant 1$.

*Proof.* For every $N$, the random variable $G'_{\Lambda_N}(\beta, s_b)$ is finite, strictly positive and $\mathcal{F}_N$-measurable with $EG'_{\Lambda_N}(\beta) = 1$. Denoting by $J_{b^{(N\setminus N-1)}}$ the variables indexed by the bonds of the 'surface' $\Lambda_N \setminus \Lambda_{N-1}$, and remarking their independence with respect to $\mathcal{F}_{N-1}$ we have that

$$\begin{aligned} E(G'_{\Lambda_N}(\beta, s_b)|\mathcal{F}_{N-1}) &= E\left(\frac{\exp(\beta \sum_{b^{(N\setminus N-1)}} J_{b^{(N\setminus N-1)}} s_{b^{(N\setminus N-1)}})}{\exp(\frac{\beta^2}{2}(2p))} G'_{\Lambda_{N-1}}(\beta, s_b)|\mathcal{F}_{N-1}\right) \\ &= G'_{\Lambda_{N-1}}(\beta, s_b)\,. \end{aligned}$$

By the martingale convergence theorem [8] we have that almost surely $\lim_{N\to\infty} G'_{\Lambda_N}(\beta, s_b) = G'_\infty(\beta, s_b)$, and, by Fatou's lemma, that $EG'_\infty(\beta) \leqslant 1$. □

Returning now to our model we can easily see that the partition function $Z_{\Lambda_N}(\beta)$ can be expressed as

$$Z_{\Lambda_N}(\beta) = \exp\left(-\frac{\beta^2}{2}(2p)^N\right) \sum_{s_b} G'_{\Lambda_N}(\beta, s_b)\,.$$

In the following section we shall study the convergence of the martingale $G_{\Lambda_N}(\beta) \equiv \frac{Z_{\Lambda_N}(\beta)}{EZ_{\Lambda_N}(\beta)}$.

## 3. High-temperature behaviour

In this section, we start by giving a condition for the uniform integrability of the martingale $G_{\Lambda_N}(\beta)$.

*Theorem 3.1.* For $0 < \beta < \sqrt{\frac{2\log 2}{p}}$, the sequence $G_{\Lambda_N}(\beta)$ converges in $L^1$ to the limit $G_\infty(\beta)$ as $N \to \infty$.

*Proof.* From proposition 2.1 we have that the limit of the martingale satisfies $EG_\infty(\beta) \leqslant 1$. We need only to prove that if $0 < \beta < \sqrt{\frac{2\log 2}{p}}$, $EG_\infty(\beta) > 0$ or equivalently that $EG_\infty^{1/2}(\beta) \neq 0$. Let us remark that one can express the martingale $G_{\Lambda_N}(\beta)$ using the following functional relation:

$$G_{\Lambda_N}(\beta) = \prod_{j=1}^{2p} G_{\Lambda_{N-1}}^{(j)}(\beta)$$

where the variables $G_{\Lambda_{N-1}}^{(j)}(\beta)$ have the same distribution as $G_{\Lambda_{N-1}}(\beta)$. Using this expression one can check that if $\beta < \sqrt{2\frac{\log 2}{p}}$, then $EG_\infty^{1/2}(\beta) > 0$ and the result follows. □

The previous theorem implies the following:

*Corollary 3.2.* If $0 < \beta < \sqrt{\frac{2\log 2}{p}}$, we have for every $N$

$$G_{\Lambda_N}(\beta) = E(G_\infty(\beta)|\mathcal{F}_N).$$

*Proof.* The assertion follows from the $L^1$ continuity of the conditional expectations:

$$E(G_{\Lambda_L}(\beta)|\mathcal{F}_N) \to E(G_\infty(\beta)|\mathcal{F}_N) \qquad \text{as} \quad L \to \infty.$$

But $E(G_{\Lambda_L}(\beta)|\mathcal{F}_N) = G_{\Lambda_N}(\beta)$ for $L \geqslant N$ and the corollary follows. □

*Definition 3.3.* The critical temperature $\beta_c$ of the model is defined as $\beta_c = \sqrt{\frac{2\log 2}{p}}$.

As a consequence of the theorem 3.1 we have the following:

*Theorem 3.4.* For $\beta < \beta_c$ the $\lim_{N\to\infty}(2p)^{-N}\log G_{\Lambda_N}(\beta)$ exists almost surely and in the mean it is equal to zero.

*Proof.* The almost sure existence of the positive limit $G_\infty(\beta)$ guarantees the almost sure existence of the $\log G_\infty(\beta)$. Moreover, $\lim_{N\to\infty}(2p)^{-N}\log G_{\Lambda_N}(\beta) = 0$. For the second assertion of the theorem, it suffices to check, using Jensen's inequality, that

$$-\infty < E\log G_\infty(\beta) \leqslant E\log G_{\Lambda_N}(\beta) \leqslant 0.$$ □

The derivation of the free energy is now given by

*Theorem 3.5.* For $0 < \beta < \beta_c$, the limit

$$-\beta f_\infty(\beta) = \lim_{N\to\infty}(2p)^{-N}\log Z_{\Lambda_N}(\beta)$$

exists almost surely. Moreover, $-\beta f_\infty(\beta) = \frac{\beta^2}{2} + \log 2$.

The previous result formulates the existence and the self-averaging property of the free energy. This property is expressed here in the strongest form. It is indeed an easy matter to check that the theorem guarantees the coincidence of the annealed $\lim_{N\to\infty}(2p)^{-N}\log EZ_{\Lambda_N}(\beta)$ and quenched $\lim_{N\to\infty}N^{-1}E\log Z_{\Lambda_N}(\beta)$ free energies. The strong self-averaging property under this form arises in the high-temperature phase of many mean-field models [1] but its general formulation is given by [19]

$$E(f_{\Lambda_N}(\beta) - Ef_{\Lambda_N}(\beta))^2 = o(|\Lambda_N|^{-1}).$$

Let us also note that we can obtain various boundary conditions by fixing the two extreme spins $s$ and $s'$ of the lattice. We have:

*Theorem 3.6.* For every temperature $\beta$ the free energy is independent of the boundary conditions.

We now can investigate the Gibbs distributions defined in section 2.

*Theorem 3.7.* For $0 < \beta < \beta_c$, the sequence of random probability measures $\nu_{\Lambda_N,\beta}(\cdot)$ converges almost surely as $N \to \infty$, to a unique limit $\nu_\beta(\cdot)$ on the Borel field of $[0, 1]$.

*Proof.* The assertion follows using theorem 3.1 and the Kolmogorov's extension theorem. □

One might ask if the previously defined random state has a Gibbsian description in the sense of the Dobrushin, Lanford and Ruelle (DLR) formalism, prescribing the finite-volume conditional distributions. As in the case of multiplicative chaos [7], it is not obvious how to carry out a version of the DLR program. It should be remarked that here we have defined and studied laws on finite partitions whose the precise relation with prescribed probability kernels is not clear.

We can, however, complete the study of the state $\nu_\beta(\cdot)$ by remarking that it is supported by the whole interval $[0, 1]$ showing that there exists almost surely a Borel set of Haussdorff dimension $1 - \frac{p\beta^2}{2\log 2}$ supporting the limit $P^\beta_\infty(\cdot)$ [12].

*Theorem 3.8.* For $0 < \beta < \beta_c$, the Hausdorff dimension $D_H(\nu_\beta)$ of the support of the measure $\nu_\beta(\cdot)$ equals $1 - \frac{p\beta^2}{2\log 2}$.

## 4. Low-temperature behaviour

In this section, we shall estimate the macroscopic free energy in the low-temperature $(\beta \geqslant \beta_c)$ region.

As a consequence of the theorem 3.1 we have the following:

*Corollary 4.1.* For $\beta \geqslant \beta_c$, the martingale $G_{\Lambda_N}(\beta)$ converges to zero as $N \to \infty$.

As we shall see, $G_{\Lambda_N}(\beta)$ is tending to zero exponentially fast and the free energy will be given by the exponential rate. The main object of the study will be the following sequence of finite volume quenched pressures:

$$P_{\Lambda_N}(\beta) = (2p)^{-N} \log Z_{\Lambda_N}(\beta)\,.$$

The definition of the partition function in section 2 implies that the above quantities are Lipschitz and convex functions of $\beta$. These properties characterize also any accumulation point of the sequence $P_{\Lambda_N}(\beta)$. We shall now formulate an upper and a lower bound for the accumulation points.

*Proposition. 4.2.* Let $\beta > \beta' > 0$. For every quenched accumulation point $P(\beta)$ of $P_{\Lambda_N}(\beta)$ we have almost surely

$$P(\beta) \leqslant P(\beta') + \beta'^2 \left(\frac{\beta}{\beta'} - 1\right).$$

*Proof.* Let $E_N$ be the set defined by

$$E_N = \left\{Z_{\Lambda_N}(\beta) \leqslant e^{\sqrt{N}\beta\beta'(\sqrt{N}+1/\beta'^2)}\right\}.$$

One can easily check using Chebyshev's inequality that $\mathrm{Prob}E_N \geqslant 1 - e^{-N}$. On the other hand, by the Borel–Cantelli lemma we have that $\mathrm{Prob}(\liminf E_N) = 1$. We shall assume that the set $\liminf E_N$ is the quenched randomness set. Observing now that on $E_N$

$$\frac{Z_{\Lambda_N}(\beta)}{e^{N\beta\beta'}} \leqslant Z_{\Lambda_N}(\beta')e^{-N\beta'^2}e^{N(\beta/\beta'-1)}$$

the result follows. □

The lower bound is given by

*Proposition. 4.3.* Let $\beta > \beta_c$. We have almost surely that

$$P(\beta) \geqslant \beta\beta_c - \tfrac{1}{2}\beta_c^2 .$$

*Proof.* Assume that for $\beta \leqslant \beta_c$, $P(\beta) = \frac{\beta^2}{2} + \log 2$. From the convexity of the function $P(\beta)$ we have that its graph must not be less than the tangent to $\frac{\beta^2}{2}$ at the critical value $\beta = \beta_c$. Hence,

$$P(\beta) \geqslant \tfrac{1}{2}\beta_c^2 + \log 2 + (\beta - \beta_c)\beta_c = \beta\beta_c - \tfrac{1}{2}\beta_c^2 + \log 2 .$$ □

We can now state the main result of this section.

*Theorem 4.4.* For $\beta > \beta_c$, the limit

$$-\beta f_\infty(\beta) = \lim_{N\to\infty} (2p)^{-N} \log Z_{\Lambda_N}(\beta)$$

exists almost surely and equals $\beta\beta_c - \beta_c^2/2 + \log 2$.

*Proof.* The assertion follows from the previous results. Namely, by applying proposition 4.2 for $\beta' = \beta_c$ and remarking that all accumulation points are equal. □

For the ground-state energy $\epsilon_0$, we obtain, letting $\beta \to \infty$, the following value:

$$\epsilon_0 = \beta_c = \sqrt{\frac{2\log 2}{p}} .$$

One can easily see that the low-temperature thermodynamic quantities (ground-state energy, entropy, etc) we obtain are very similar to all the mean-field models (REM, GREM, polymers on disordered trees, ... [9, 10, 4]).

One can define the low-temperature random states by the limits on the subsequences of $\nu_{\Lambda_N,\beta}(\cdot)$ for $\beta > \beta_c$. The study of their structure will be studied elsewhere. From the previous results we can, however, conclude that these limits are supported by many points of the unit interval.

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